Assignment 9

1. Using Euler's method, approximate $y(1)$ and $z(1)$ with $h = 0.5$ and again with $h = 0.25$ for the initial-value problem defined by

$$y^{(1)}(t) = 2y(t) + z(t) + t - 1$$
$$z^{(1)}(t) = y(t) - 2z(t) - t - 2$$
$$y(0) = 1$$
$$z(0) = 2$$

```
f = @(t, w)( [2*w(1) + w(2) + t - 1; w(1) - 2*w(2) - t - 2] );
w0 = [1 2]';
h = 0.5;
ts = 0:h:1;
ws = zeros( 2, 3 );
ws(:, 1) = w0;
for k = 1:2
    ws(:, k+1) = ws(:, k) + h*f(ts(k), ws(:,k));
end
ws
    ws = 1.0   2.5   4.5
         2.0  -0.5   0.0


h = 0.25;
tss = 0:h:1;
wss = zeros( 2, 5 );
wss(:, 1) = w0;
for k = 1:4
    wss(:, k+1) = wss(:, k) + h*f(tss(k), wss(:,k));
end
wss
     wss = 1.0   1.75   2.625   3.875     5.7890625
           2.0   0.75   0.25    0.15625   0.359375
```

Important: Note how too large a step size introduced a completely erroneous approximate of $z(0.5)$ when $h = 0.5$? The approximation was $z(0.5) \approx -0.5$, which is negative, even though when we used $h = 0.25$, we found the approximation $z(0.5) \approx 0.25$.

2. In Question 1, approximate $y(0.1)$ and $z(0.1)$ using one step of Euler's method.

```
f = @(t, w)( [2*w(1) + w(2) + t - 1; w(1) - 2*w(2) - t - 2] );
t0 = 0;
w0 = [1 2]';
h = 0.1;
w1 = w0 + h*f(t0, w0)
      w1 = 1.3
           1.5
```

3. In Question 1, approximate $y(0.1)$ and $z(0.1)$ using one step of Heun's method.

```
f = @(t, w)( [2*w(1) + w(2) + t - 1; w(1) - 2*w(2) - t - 2] );
w0 = [1 2]';
t0 = 0;
h = 0.1;
s0 = f(t0, w0)
      s0 =  3
           -5
s1 = f(t0 + h, w0 + h*s0)
      s1 =  3.2
           -3.8
w1 = w0 + h*(s0 + s1)/2.0
      w1 =  1.31
            1.56
```

4. In Question 1, approximate $y(0.1)$ and $z(0.1)$ using one step of the 4^{th}-order Runge-Kutta method.

```
f = @(t, w)( [2*w(1) + w(2) + t - 1; w(1) - 2*w(2) - t - 2] );
w0 = [1 2]';
t0 = 0;
h = 0.1;
s0 = f(t0, w0)
      s0 =  3
           -5
s1 = f(t0 + h/2, w0 + h/2*s0)
      s1 =  3.1
           -4.4
s2 = f(t0 + h/2, w0 + h/2*s1)
      s2 =  3.14
           -4.455
s3 = f(t0 + h,   w0 + h  *s2)
      s3 =  3.2825
           -3.895
w1 = w0 + h*(s0 + 2*s1 + 2*s2 + s3)/6.0
      w1 = 1.312708333333333
           1.556583333333333
```

5. Given the results in Questions 2 and 3, what is the value of *a* for the adaptive Euler-Heun method for this one step if $\varepsilon_{abs} = 0.1$? What would be the next value of *h* used, and would you be recalculating the first point, or would you be calculating the value at $t = 0.1 + h$ with the new value of *h*?

```
y = [1.3 1.5]';
z = [1.31 1.56]';
eps_abs = 0.1;
h = 0.1;
a = (h*eps_abs)/(2*norm(y - z))
        a = 8.219949365267860e-02
h = 0.9*a*h
        h = 7.397954428741075e-03
```

You would need to recalculate with half the previous step size. Note that you may end up halving multiple times, but this would only be for the first step—a smaller initial *h* should have been chosen (less than 0.1).

6. Convert the following 3rd-order initial-value problem into a system of 1st-order initial value problems:

$$\begin{aligned} y^{(3)}(t) &= 2y(t) + y^{(1)}(t) + 0.5y^{(2)}(t) + t - 1 \\ y(0) &= 1.2 \\ y^{(1)}(0) &= 1.3 \\ y^{(2)}(0) &= 1.4 \end{aligned}$$

This would result in

$$\mathbf{w}(t) = \begin{pmatrix} w_0(t) \\ w_1(t) \\ w_2(t) \end{pmatrix} = \begin{pmatrix} y(t) \\ y^{(1)}(t) \\ y^{(2)}(t) \end{pmatrix} \text{ so } \mathbf{w}^{(1)}(t) = \begin{pmatrix} w_0^{(1)}(t) \\ w_1^{(1)}(t) \\ w_2^{(1)}(t) \end{pmatrix} = \begin{pmatrix} y^{(1)}(t) \\ y^{(2)}(t) \\ y^{(3)}(t) \end{pmatrix} = \begin{pmatrix} w_1(t) \\ w_2(t) \\ 2w_0(t) + w_1(t) + 0.5w_2(t) + t - 1 \end{pmatrix} \text{ with } \mathbf{w}(0) = \begin{pmatrix} 1.2 \\ 1.3 \\ 1.4 \end{pmatrix}$$

7. Convert the following system of three 2^rd^-order initial-value problem into a system of 1^st^-order initial value problems.

$$x^{(2)}(t) = 2x(t)y^{(1)}(t) - 1.2$$
$$y^{(2)}(t) = 4y(t)z^{(1)}(t) - 1.7$$
$$z^{(2)}(t) = 3z(t)x^{(1)}(t) - 1.9$$
$$x(0) = 1.4$$
$$x^{(1)}(0) = -1.5$$
$$y(0) = 1.6$$
$$y^{(1)}(0) = -1.9$$
$$z(0) = 0.1$$
$$z^{(1)}(0) = -1.3$$

This would result in

$$\mathbf{w}(t) = \begin{pmatrix} w_0(t) \\ w_1(t) \\ w_2(t) \\ w_3(t) \\ w_4(t) \\ w_5(t) \end{pmatrix} = \begin{pmatrix} x(t) \\ x^{(1)}(t) \\ y(t) \\ y^{(1)}(t) \\ z(t) \\ z^{(1)}(t) \end{pmatrix} \text{ so } \mathbf{w}^{(1)}(t) = \begin{pmatrix} w_0^{(1)}(t) \\ w_1^{(1)}(t) \\ w_2^{(1)}(t) \\ w_3^{(1)}(t) \\ w_4^{(1)}(t) \\ w_5^{(1)}(t) \end{pmatrix} = \begin{pmatrix} x^{(1)}(t) \\ x^{(2)}(t) \\ y^{(1)}(t) \\ y^{(2)}(t) \\ z^{(1)}(t) \\ z^{(2)}(t) \end{pmatrix} = \begin{pmatrix} w_1(t) \\ 2w_0(t)w_3(t) - 1.2 \\ w_3(t) \\ 4w_2(t)w_5(t) - 1.7 \\ w_5(t) \\ 3w_4(t)w_1(t) - 1.9 \end{pmatrix} \text{ with } \mathbf{w}(0) = \begin{pmatrix} 1.4 \\ -1.5 \\ 1.6 \\ -1.7 \\ 0.1 \\ -1.3 \end{pmatrix}$$

8. Use three steps of the shooting method to approximate a solution to the boundary-value problem defined by

$$u^{(2)}(x) = 0.3u^{(1)}(x) + 0.1u(x) - x - 0.2$$
$$u(0) = 2$$
$$u(1) = 3$$

At each step of the shooting method, you will use Euler's method with $h = 0.2$.

```
% This is the set-up for our second-order ODE, which is
% converted into a system of two first-order ODEs:
f = @(x, w)( [w(2); 0.3*w(2) + 0.1*w(1) - x - 0.2] );
h = 0.2;
xs = 0:h:1;
a = 0;
u_a = 2;
b = 1;
u_b = 3;

% This is our initial guess at a slope
s0 = (u_b - u_a)/(b - a)
     s0 = 1

% We are applying Euler's method with h = 0.2, so we will approximate
% the solution at 0.2, 0.4, 0.6, 0.8 and 1.0, and at 0, we have the two
% initial conditions u(a) = 2 and u'(a) = s0
us = [u_a 0 0 0 0 0
      s0  0 0 0 0 0];

% Run Euler's method
for k = 1:5
  us(:, k + 1) = us(:, k) + h*f(xs(k), us(:, k));
end

% Here are the approximations
%  - Recall, the first row is y(0) and approximations of y(0.2), y(0.4), ...
%    and the second row is y'(0) and approximations of y'(0.2), y'(0.4), ...
us
     us = 2.00000   2.20000   2.41200   2.62952   2.84574   3.05345
          1.00000   1.06000   1.08760   1.08110   1.03855   0.95778

% The estimate of u(b) with this approximation is the last
% entry of the first row:
u0b = us(1, end)
    u0b = 3.0534
```

```
% Now we calculate the second slope to use, by correcting for the
% error of the first approximation
%  - Recall that we overshot, so this slope will be slightly less
s1 = (2*u_b - us(1, end) - u_a)/(b - a)
     s1 = 0.94655

% Similar set-up as above, and run Euler's method:
us = [u_a 0 0 0 0 0
      s1  0 0 0 0 0];

for k = 1:5
  us(:, k + 1) = us(:, k) + h*f(xs(k), us(:, k));
end

% You will see that this time, we undershot:
%  - We wanted to find an initial slope such that y(b) = 3.0
us
     us = 2.00000   2.18931   2.38998   2.59544   2.79880   2.99274
          0.94655   1.00334   1.02733   1.01677   0.96968   0.88384

% Here is our estimate of y(b) with the initial slope s1
u1b = us(1, end);
    u1b =  2.9927

% Now that we have two initial slopes, we now apply the secant
% method on the root-finding
%
%              u (b) - u  = 0
%               s       b
%
% where this is an expression in the initial slope 's'
s2 = (s0*(u1b - u_b) - s1*(u0b - u_b))/((u1b - u_b) - (u0b - u_b))
     s2 =  0.95295

% Repeat...
us = [u_a 0 0 0 0 0
      s2  0 0 0 0 0];

for k = 1:5
  us(:, k + 1) = us(:, k) + h*f(xs(k), us(:, k));
end

us
     us = 2.00000   2.19059   2.39261   2.59952   2.80442   3.00000
          0.95295   1.01012   1.03454   1.02447   0.97792   0.89269
```

You will note that $u(0) = 2$ and $u(1) = 3$, so we have found an approximation of a solution to this BVP, and thus approximations are $u(0.2) = 2.19059$, $u(0.4) = 2.39261$, $u(0.6) = 2.59952$ and $u(0.8) = 2.8442$.

9. The actual solution to the boundary-value problem given in Question 8 is

$$u(x) = 3\frac{e^{-0.2x}\left(10e^{0.5} - 7\right) - e^{0.5x}\left(10e^{-0.2} - 7\right)}{e^{0.5} - e^{-0.2}} + 10x - 28.$$

How close is your approximation to this solution at x = 0.2, 0.4, 0.6 and 0.8?

We can define

```
u = @(x)( 3*(
    exp(-0.2*x)*(10*exp(0.5) - 7) - exp(0.5*x)*(10*exp(-0.2) - 7)
)/(exp(0.5) - exp(-0.2)) + 10*x - 28 );
```

Thus, we see that the errors are:

```
>> us( 1, : ) - u( 0:0.2:1 )
   ans = -3.5527e-15  -0.013478  -0.020774  -0.021381  -0.014694  -3.5527e-15
```

The errors associated with $u(0)$ and $u(1)$ (the boundary values) are a result of numeric errors in the calculations of $u(x)$ at 0 and 1, respectively; however, the errors of the approximations at the intermediate points are due to the errors in the approximation due to using Euler's method. The maximum value of the 2nd derivative on the interval [0, 1] of the solution is 0.3, and thus, the maximum error should be less than the error given by the approximation $\frac{1}{2}(b-a)h\, u^{(2)}(\xi)$ where $0 \le \xi \le 1$ and the maximum of the 2nd derivative occurs at $x = 0$ when $u^{(2)}(0) = 0.3$, so $\frac{1}{2}\cdot 1\cdot 0.2\cdot 0.3 = 0.03$, and we note all the errors in absolute value are less than this.

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